Given: Sky diver with $m = 80\text{ kg}$ falling straight down. 
Drag force is $F_d = kv^2$; $k = 0.27 \text{ N} \cdot \text{s}^2/\text{m}^2$; $v$ is speed.

Find: (a) Terminal speed, $v_t$, of sky diver.
(b) Vertical distance to reach $v = 0.95v_t$.
(c) Vertical distance to reach same speed if $F_d = 0$.

Plot: (a) Speed $v = v(t)$, and (b) $v = v(s)$.

Solution:
Consider sky diver as a system; apply Newton's 2nd law of motion.

Basic equation: $\Sigma F_s = m \frac{dv}{dt} = mv \frac{dv}{ds} = mg - F_d$ (1)

Assumptions: Initial condition $v = 0$ at $t = 0$, $s = 0$.

Then $\Sigma F_y = mg - kv^2 = ma$.

At terminal speed $a = 0$ and $v = v_t$. Thus:
$$mg - kv_t^2 = 0 \quad \text{and} \quad v_t = \sqrt{\frac{mg}{k}}$$ (2)

To solve for $s$ at $v = 0.95v_t$, we need an expression relating $v$ and $s$. From Eq. (1),
$$mg - kv^2 = mv \frac{dv}{ds} \quad \text{or} \quad 1 - \frac{v^2}{v_t^2} = \frac{v}{v_t} ds \quad \text{or} \quad \frac{v}{v_t} ds = 1 - \frac{v^2}{v_t^2}$$

Separating variables and integrating,
$$\int_{0}^{s} \frac{v}{v_t} ds = \int_{0}^{v} \frac{g}{v_t} \frac{dv}{1 - \frac{v^2}{v_t^2}}$$

$$\frac{1}{2} \ln \left[ \frac{1 - \frac{v^2}{v_t^2}}{1 - \frac{v^2}{v_t^2}} \right] = g \frac{v}{v_t} \quad \text{or} \quad s = \frac{1}{2} \left( \frac{1}{v^2} \right)$$ (3)

For $v = 0.95v_t$, then
$$s = \frac{1}{2} \times 80 \text{ kg} \times 0.27 \text{ N} \cdot \text{s}^2/\text{m}^2 \ln \left[ \frac{1 - (0.95)^2}{1 - (0.95)^2} \right] \frac{1\text{ m}}{\text{kg} \cdot \text{m}^2/\text{s}^2} = 345 \text{ m} \quad \text{s}$$

(c) For free fall without air resistance $\Sigma F_s = mg = mv \frac{dv}{ds}$

Integrating we obtain $\frac{v^2}{2} = gs$ or
$$s = \frac{1}{2} v^2$$
Evaluating, we obtain
\[ s = \frac{1}{2} \times [0.95(53.9)^{2}] \times 9.81 \, \text{m/s}^2 = 134 \, \text{m} \]

Thus, our resistance significantly increases the vertical fall distance needed to reach a given speed.

From Eq. 3, we can write \( V = V(s) \) as
\[ V = \frac{1}{2} \left[ 1 - e^{-2kt \sin \theta} \right] \]

To obtain an expression for \( V = V(t) \), we write
\[ \sum F = mg - kV = \frac{dv}{dt} \]

Separating variables and integrating, we have
\[ \int_0^t \, dt = \int_0^V \frac{m \, dv}{mg - kV} = -\frac{1}{k} \ln \left( \frac{V}{V_0} + 1 \right) \]

Then
\[ t = \frac{1}{k} \ln \left[ \frac{V_0}{V_0 + V} + \frac{V}{V_0 + V} \right] = \frac{1}{k} \ln \left( \frac{V_0 + V}{V_0} + 1 \right) \]

and
\[ \frac{V + V_0}{V_0} = e^{\frac{kV_0}{t} t} \]

Finally
\[ \frac{V}{V_0} = \frac{e^{\frac{kV_0}{t} t} - 1}{e^{\frac{kV_0}{t} t} + 1} = \tanh \left( \frac{kV_0}{t} t \right) \]

Eqs. 4 and 5 are plotted below.

**Eq. 4: Speed Ratio vs. Distance**

**Eq. 5: Speed Ratio vs. Time**