Ronge-Kutta Method

Consider the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0. \]  

(1)

(2)

We look for a solution in the form

\[ y(x + h) = y(x) + ak_1 + bk_2, \]  

(3)

where

\[ k_1 = hf(x, y), \]  

(4)

and

\[ k_2 = hf(x + ah, y + \beta k_1). \]  

(5)

Combining (3)-(5) gives

\[ y(x + h) = y(x) + ahf(x, y) + bhf(x + ah, y + \beta hf(x, y)). \]  

(6)

However

\[ f(x + ah, y + \beta hf(x, y)) = f(x, y) + ahf_x(x, y) + bhf_y(x, y)f_y(x, y), \]  

(7)

and hence (6) can be written in the form

\[ y(x + h) = y(x) + ahf(x, y) + bhf(x, y)f_y(x, y) + \beta hf(x, y)f_y(x, y) \]  

(8)

or equivalently

\[ y(x + h) = y(x) + (a + b)hf(x, y) + h^2(ahf_x(x, y) + \beta hf(x, y)f_y(x, y)) \]  

(9)

Taylor series expansion gives

\[ y(x + h) = y(x) + hf(x, y) + \frac{h^2}{2} f'(x, y), \]  

(10)

by virtue of (1), and by the chain rule we have
\[ \frac{df(x, y)}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = f_x + f_y f. \]  

(11)

Hence (10) yields

\[ y(x + h) = y(x) + hf(x, y) + \frac{h^2}{2} \left( f_x(x, y) + f_y(x, y)f(x, y) \right). \]  

(12)

Comparing (9) to (12) we obtain

\[ a + b = 1, \]

(13)

\[ \alpha b = \frac{1}{2}, \]

(14)

and

\[ \beta b = \frac{1}{2}. \]

(15)

Equations (13) to (14) form three equations with four unknowns. Taking \( b = 1/2 \), for example, gives \( a = 1/2, \alpha = 1 \), and \( \beta = 1 \). So for this case the second-order Runge-Kutta scheme gives

\[ k_1 = hf(x, y), \]

(16)

\[ k_2 = hf(x + h, y + k_1), \]

(17)

\[ y(x + h) = y(x) + \frac{1}{2} k_1 + \frac{1}{2} k_2, \]

(18)

which is equivalent to the modified Euler method

\[ y'(x) = f(x, y), \]

(19)

\[ y'(x + h) = f(x + h, y + hf(x, y)), \]

(20)

\[ y(x + h) = y(x) + h \frac{y'(x) + y'(x + h)}{2}. \]

(21)
Fourth-order Runge-Kutta

\[ y(x + h) = y(x) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \]

\[ k_1 = hf(x, y) \]

\[ k_2 = hf\left( x + \frac{h}{2}, y + \frac{k_1}{2} \right) \]

\[ k_3 = hf\left( x + \frac{h}{2}, y + \frac{k_2}{2} \right) \]

\[ k_4 = hf(x + h, y + k_3) \]