Review of dynamics

1.a. Dynamics of Particles

**Fundamental properties:** \( m, t, r, f \) (mass, time, position, force)

**Definitions**
\[
\begin{align*}
v & = \frac{dr}{dt}, \\
a & = \frac{dv}{dt}.
\end{align*}
\]

**Axiomatic laws** (for particles):
\[ f = ma \] (1.3)

and for particles in contact
\[ f_{ij} = -f_{ji} \] (1.4)

where \( f_{ij} \) is the force that particle \( i \) applies on \( j \).

**Work** done by variable force \( f \) acting on a particle \( P \) which moving from \( s_1 \) to \( s_2 \) along the trajectory \( S \) is defined by
\[ W_{1-2} \triangleq \int_{s_1}^{s_2} f \cdot ds \] (1.5)

The force is conservative if there exists a potential function \( V(r) \) such that
\[ W_{1-2} = V(r_1) - V(r_2). \] (1.6)

The work in this case is independent of the path of motion.

The kinetic energy for the particle is defined by
\[ T \triangleq \frac{1}{2} mv^2 \] (1.7)

and it follows from (1.3) that
\[ W_{1-2} = T_2 - T_1. \] (1.8)

The momentum of the particle is defined by \( mv \), and it also follows from (1.3) that
\[ \int_{t_i}^{t_f} f dt = m v_2 - m v_1. \] (1.9)

1.b. Dynamics of Rigid Bodies

**Rigid body** is a system of particles with center of gravity \( G \). Its angular acceleration \( \omega \) is a vector in the direction of its instantaneous axis of rotation, with magnitude that is equal to the angular speed of rotation with positive counterclockwise sense.
It follows from (1.3) & (1.4) that for rigid body in plane motion
\[ \mathbf{f} = m \mathbf{a}_G \]  
(1.10)
\[ \mathbf{M}_G = I_G \alpha \]  
(1.11)
where the moment \( \mathbf{M} \) is defined by
\[ \mathbf{M}_G = \mathbf{r} \times \mathbf{f} \]  
(1.12)
where \( \mathbf{r} \) is any vector from \( G \) to the line of action of \( \mathbf{f} \), and the angular acceleration \( \alpha \) is
\[ \alpha = \lim_{\Delta t \to 0} \frac{\omega(t + \Delta t) - \omega(t)}{\Delta t} \]  
(1.13)
The kinetic energy for rigid body is given by
\[ T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 \]  
(1.14)
and (1.8) holds.
The angular momentum for rigid body is defined by \( I_G \omega \), and it follows from (1.11) that
\[ \int_{t_1}^{t_2} \mathbf{M}_G dt = I_G \omega_2 - I_G \omega_1 \]  
(1.15)

1.c. Relative Motion

The Operator rule for differentiation. Let \( XYZ \) be a stationary system and let \( xyz \) be a moving coordinate system with variable angular velocity \( \omega \). Then for any vector \( \mathbf{h} \)
\[ \left( \frac{d\mathbf{h}}{dt} \right)_{XYZ} = \left( \frac{d\mathbf{h}}{dt} \right)_{xyz} + \omega \times \mathbf{h} \]  
(1.16)
where \( \left( \frac{d\mathbf{h}}{dt} \right)_{XYZ} \) is the time derivative of \( \mathbf{h} \) in terms of the stationary coordinate system and \( \left( \frac{d\mathbf{h}}{dt} \right)_{xyz} \) is the time derivative of \( \mathbf{h} \) done by an observer which is attached to the rotating coordinates \( xyz \) (and suffers therefore from a severe headache).
Let $O$ and $A$ be the origins of $XYZ$ and $xyz$ respectively. Let $\mathbf{r}_A$ and $\mathbf{r}_B$ be the vector position of $A$ and a particle $B$ in terms of $XYZ$ as shown in the figure below. Then

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \quad (1.17)$$

where $\mathbf{r}_{B/A}$ is the position of $B$ as observed by an observer at $A$. Application of the operator rule gives

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} + \omega \times \mathbf{r}_{B/A} \quad (1.18)$$

and

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} + \omega \times (\omega \times \mathbf{r}_{B/A}) + 2\omega \times \mathbf{v}_{B/A} + \omega \times \mathbf{r}_{B/A} \quad (1.19)$$

where $\mathbf{v}_{B/A}$ and $\mathbf{a}_{B/A}$ are the velocity and acceleration of $B$ as observed by the observer at $A$.

**Remark.** The above equations describe the Galilean Transformation. According to this transformation, contrary to experimental results, the speed of light $c$ in $XYZ$ is different from that observed in $xyz$. Let the position of $A$ as a function of time $t$ be $\mathbf{v}t \mathbf{k}$ where $v$ is constant, and suppose that $X$ and $Y$ are parallel to $x$ and $y$ respectively for all time $t$ (i.e. $\omega = 0$ for this case). Then the Lorentz Transformation is

$$
\begin{pmatrix}
\mathbf{r}_{B_x} \\
\mathbf{r}_{B_y} \\
\mathbf{r}_{B_z}
\end{pmatrix} =
\begin{pmatrix}
\mathbf{r}_{B/A_x} \\
\mathbf{r}_{B/A_y} \\
\mathbf{r}_{B/A_z} \sqrt{1 - \beta^2} + vt
\end{pmatrix},
$$

where $\beta = v/c$.

The time in the two systems is not the same

$$t_A = \frac{t - \frac{\mathbf{v}\mathbf{r}_{A/A}}{c^2}}{\sqrt{1 - \beta^2}}.
$$

We will stick however to the Newtonian dynamics. For small $\beta$ the two transformations give almost identical results.
**Bonus Problem 1.1.**

We read in Whittaker\(^1\), p.2:

Consider a rigid body, one of whose points is made immoveable by some attachment; suppose that the body is free to turn about this point in any manner, and let any two possible configurations of the body be taken: for convenience we shall call these the configuration \( P \) and the configuration \( Q \). We shall now shew that it is possible to bring the body from the configuration \( P \) to the configuration \( Q \) by simply rotating it about some definite line through the fixed point, i.e. that a rotation about a point is always equivalent to a rotation about a line through the point.

So suppose that three unit vectors defining an orthogonal coordinate system are initially (configuration \( P \)) in the directions

\[
\begin{pmatrix}
1 \\
0 \\
0 
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0 
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1 
\end{pmatrix},
\]

and that the system is then rotated about its origin such that at the new position (configuration \( Q \)) the unit vectors are in the directions

\[
\begin{pmatrix}
1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \\
2/\sqrt{6} & 1/\sqrt{3} & 0 \\
1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} 
\end{pmatrix}
\]

**Find:** (a) the axis of finite rotation from \( P \) to \( Q \), and (b) the angle that the system is rotated while moving from \( P \) to \( Q \).

*E-mail the results (include in your message only three vector components and one angle)*

to: ram@me.lsu.edu

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\(^1\) E.T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies with an introduction to the problem of three bodies*, Cambridge University Press, London, 1937