Energy methods

2.a. Energy Method (Single Degree of Freedom System)

Another possible way to determine the dynamic of a system is by starting with the same fundamental properties $m$, $t$, $r$, $f$, defining the velocity, kinetic energy and potential energy as in (1.1), (1.7) and (1.6), respectively, together with the action $L$

$$L(t, x, \dot{x}) = T - V,$$  \hspace{1cm} (2.1)

and replace the axiomatic law (1.3) by the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0,$$  \hspace{1cm} (2.2)

which ensures that the action is stationary.

2.b. Examples

(i) Mass-spring system

Consider the system shown below. Let $x(t)$ be the displacement of the mass from its static equilibrium. Then it follows from the free-body diagram shown and (1.3) that

$$m\ddot{x} + kx = 0.$$  \hspace{1cm} (2.3)

For this case

$$T = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2$$

and

$$L = \frac{1}{2}(m\dot{x}^2 - kx^2),$$

and (2.2) gives

$$-kx - \frac{d}{dt}m\dot{x} = 0,$$

from which we obtain (2.3).

(ii) The Simple Pendulum

Consider the simple pendulum shown below. Then (1.3) applied in the $A$-$A$ direction (perpendicular to $T$) gives

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0.$$  \hspace{1cm} (2.4)

For this case

$$T = \frac{1}{2}m(\dot{\theta})^2, \quad V = mgl(1 - \cos \theta)$$
and hence
\[ L = \frac{1}{2} m (l \dot{\theta})^2 - mgl(1 - \cos \theta), \]
and (2.2) gives
\[ -mgl \sin \theta - \frac{d}{dt} ml^2 \dot{\theta}^2 = 0, \]
from which we obtain (2.4).

(iii) The Rolling Stone
Disk B, of mass \( m \) and moment of inertia \( I_G \), rolls on circle A without slipping as shown below.

Let \( \phi \) be the angle of rotation of the disk. Then the no-slip condition implies that
\[ \dot{\theta}(R - r) = -\phi \dot{r}. \]  
(Simply express the velocity of the center of the disk \( o \) in terms of \( \theta \) and \( \phi \)).

Newton's second law (1.10) in the direction perpendicular to \( N \) gives
\[ F - mg \sin \theta = m(R - r) \dot{\phi}, \]  
and the Euler second law (1.11) gives
\[ Fr = I_G \ddot{\phi}. \]  
Differentiating (2.5) w.r.t. \( t \) and substituting the result in (2.7) gives
\[ F = - \frac{R-r}{r^2} I_g \dot{\theta} \]

so that (2.6) yields

\[ (R-r) \left( m + \frac{I_g}{r^2} \right) \ddot{\theta} + mg \sin \theta = 0 \quad (2.8) \]

The problem can be solved alternatively by the energy method. Using (2.5) we find that the total kinetic energy of disk \( B \) is

\[ T = \frac{1}{2} m (R-r)^2 \dot{\theta}^2 + \frac{1}{2} I_g \left( \frac{R-r}{r^2} \right)^2 \dot{\theta}^2 = \frac{1}{2} (R-r)^2 \left( m + \frac{I_g}{r^2} \right) \dot{\theta}^2 \]

The potential energy of disk \( B \) is

\[ V = mg(R-r)(1-\cos \theta) \]

and hence

\[ L = \frac{1}{2} (R-r)^2 \left( m + \frac{I_g}{r^2} \right) \dot{\theta}^2 - mg(R-r)(1-\cos \theta). \]

From the Euler-Lagrange equation (2.2) we have

\[ -mg(R-r)\sin \theta - \frac{d}{dt} \left( \frac{d}{dt} \left( \frac{I_g}{r^2} \right) \dot{\theta} \right) = 0 \]

which determines the equation of motion (2.8).

For uniform disk \( I_g = \frac{1}{2} mr^2 \), and for this case we obtain from (2.8)

\[ \dot{\theta} + \frac{g}{1.5(R-r)} \sin \theta = 0. \]

The center of the disk oscillates therefore like a simple pendulum of length \( 1.5(R-r) \).

Note that although friction is applied to the disk at the point of contact the system is conservative. The friction force does no work when applied to the contact point of the non-slipping disk.