Lecture 6

An Inverse Problem

5.a. Analytical Tools

i. The Cramer's Rule
Consider the set of \( n \) linear equations

\[
\mathbf{A} \mathbf{x} = \mathbf{b}
\]

where \( \mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \), \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \), \( \mathbf{b} = (b_1, b_2, \ldots, b_n)^T \).

By the Cramer's rule the \( i \)-th element of the solution is

\[
x_i = \frac{\det(\hat{\mathbf{A}})}{\det(\mathbf{A})}
\]

where \( \hat{\mathbf{A}} \) is defined as follows

\[
\hat{\mathbf{A}} = [\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n].
\]

(In words, \( \hat{\mathbf{A}} \) is the same as \( \mathbf{A} \), except that the vector \( \mathbf{b} \) replaces the \( i \)-th column \( \mathbf{a}_i \)).

ii. Eigenvalues and Eigenvectors of Symmetric Matrix
Let \( \mathbf{A} \) be a real symmetric matrix, i.e.

\[
\mathbf{A}^T = \mathbf{A}.
\]

Then there exists an orthonormal matrix \( \mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \),

\[
\mathbf{U}^T \mathbf{U} = \mathbf{U} \mathbf{U}^T = \mathbf{I}
\]

and diagonal matrix \( \Lambda \)

\[
\Lambda = \begin{bmatrix}
\lambda_1 \\
& \lambda_2 \\
& & \ddots \\
& & & \lambda_n
\end{bmatrix}
\]

such that

\[
\mathbf{A} \mathbf{U} = \mathbf{U} \Lambda.
\]

Then (5.5) implies that

\[
\mathbf{U}^T \mathbf{A} \mathbf{U} = \Lambda
\]

Alternatively we may write (5.7) in the form

\[
\mathbf{A} \mathbf{u}_i = \lambda_i \mathbf{u}_i, \; i = 1, 2, \ldots, n.
\]

Note that this is a special case of (4.19) with \( \mathbf{M} = \mathbf{I} \).

It follows from the definition of the eigenvalues that

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = \prod_{i=1}^{n}(\lambda_i - \lambda)
\]

iii. Eigenvalue-Eigenvector Relation
Consider the problem

\[
\ddot{\mathbf{x}} + \mathbf{A} \mathbf{x} = \mathbf{e}_1 \sin \omega t
\]
where $A$ is symmetric and (as usual) $e_i$ is the $i$-th unit vector. We may regard $\omega$ as a variable, similar to the time $t$. Then we have a particular solution of the form
\[ x(t, \omega) = z(\omega) \sin \omega t. \] (5.12)
Substituting (5.12) in (5.11) gives
\[ (A - \lambda I)z = e_1, \quad \lambda = \omega^2. \] (5.13)
We multiply (5.13) by $U^T$
\[ U^T (A - \lambda I) U U^{-1} z = U^T e_i \] (5.14)
and use the relations (5.5), (5.8)
\[ (A - \lambda I) U^{-1} z = U^T e_i. \] (5.15)
It follows from (5.15) that
\[ z = U (A - \lambda I)^{-1} U^T e_i. \] (5.16)
Multiplying (5.16) by $e_i^T$ gives
\[ e_i^T z = e_i^T U (A - \lambda I)^{-1} U^T e_i. \] (5.17)
Note that
\[ z_1 = e_i^T z \] (5.18)
and
\[ e_i^T U = (u_{i1}, u_{i2}, \ldots, u_{in}), \] (5.19)
i.e., $e_i^T U$ is the first row of $U$.
Substituting (5.18) and (5.19) in (5.17) gives
\[ z_1 = (u_{i1}, \ldots, u_{in}) \begin{bmatrix} (\lambda_1 - \lambda)^{-1} & 0 & \cdots & 0 \\ 0 & (\lambda_2 - \lambda)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_n - \lambda)^{-1} \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{in} \end{bmatrix} \] (5.20)
or using component form
\[ z_1 = \sum_{i=1}^{n} \frac{u_{i1}^2}{\lambda_i - \lambda}. \] (5.21)
The Cramer's rule, however, applied to (5.13) gives
\[ z_1 = \frac{\det(Z(\lambda))}{\det(A - \lambda I)}. \] (5.22)
where
\[ Z(\lambda) = [a_1 - \lambda e_1, a_2 - \lambda e_2, \ldots, a_{n-1} - \lambda e_{n-1}, e_n]. \] (5.23)
Let $\hat{A}$ be the $(n-1)\times(n-1)$ matrix obtained by deleting the first row and column of $A$. Let $\mu_i, \ i=1,2,\ldots,n-1$ be the eigenvalues of $A$. Then evaluating the determinant of $Z(\lambda)$ in (5.23) by its $n$-th column gives
\[ \det(Z(\lambda)) = \det(\hat{A} - \lambda I). \] (5.24)
Hence invoking (5.10) we find
\[ \det(Z(\lambda)) = \prod_{i=1}^{n-1} (\mu_i - \lambda). \] (5.25)
We use (5.10) again and obtain from (5.22) and (5.25)
\[
Z_1 = \prod_{i=1}^{n-1} (\mu_i - \lambda) / \prod_{i=1}^{n} (\lambda_i - \lambda).
\] (5.26)

It follows from (5.21) and (5.26) that
\[
\sum_{i=1}^{n} u_{ii}^2 = \frac{\prod_{i=1}^{n-1} (\mu_i - \lambda)}{\prod_{i=1}^{n} (\lambda_i - \lambda)}.
\] (5.27)

Multiplying (5.27) by \((\lambda_j - \lambda)\) gives
\[
u_{jj}^2 + \sum_{i=1, i \neq j}^{n} (\lambda_j - \lambda) \frac{u_{ii}^2}{\lambda_j - \lambda} = \frac{\prod_{i=1}^{n-1} (\mu_i - \lambda)}{\prod_{i=1, i \neq j}^{n} (\lambda_i - \lambda)}.
\] (5.28)

Equation (5.28) holds for all \(\lambda \neq \lambda_j\). In the limit where \(\lambda \to \lambda_j\) we therefore obtain
\[
u_{jj}^2 = \frac{\prod_{i=1}^{n-1} (\mu_i - \lambda_j)}{\prod_{i=1, i \neq j}^{n} (\lambda_i - \lambda_j)}, \quad j = 1, 2, \ldots, n.
\] (5.29)

Equation (5.29) gives us a relation between the eigenvalues and the elements of the first row of the normalized eigenvector matrix.

5.b Formulation of the Inverse Problem

Consider the mass-spring system shown below.

Its mass matrix and stiffness matrices are
\[
M = \begin{bmatrix}
m_1 & & & \\
& m_2 & & \\
& & \ddots & \\
& & & m_n
\end{bmatrix}
\] (5.30)

and
Denote the natural frequencies of this system by $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Suppose now that the mass $m_1$ is pinned to the ground, as shown in the figure below:

![Mass-Spring System Diagram](image)

Then the mass and stiffness matrices of the constrained system are

$$\hat{M} = \begin{bmatrix} m_2 & \cdots & m_n \end{bmatrix}$$

$$\hat{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & \cdots & \cdots & \cdots \\ -k_2 & k_{n-2} - k_{n-1} & -k_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_{n-1} & k_{n-1} - k_n \end{bmatrix}$$

Denote the natural frequencies of this system by $\mu_1, \mu_2, \ldots, \mu_{n-1}$.

**Problem Definition: Reconstruction of a Mass-Spring System from Eigendata**

Given: $\lambda_1, \lambda_2, \ldots, \lambda_n$, $\mu_1, \mu_2, \ldots, \mu_{n-1}$ and the total mass of the system $m_T = \sum_{i=1}^{n} m_i$.

Find: $m_1, m_2, \ldots, m_n$ and $k_1, k_2, \ldots, k_n$.

5.c The collocated sine response function at the free end

Before addressing the solution to the above problem let us investigate the response of the system to the harmonic excitation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{e}_1 \sin \omega t.$$  \hspace{1cm} (5.34)

We already know that there exists a particular solution of the form

$$\mathbf{x}(\omega, t) = \mathbf{v}(\omega) \sin \omega t$$

and that $\mathbf{v}(\omega)$ is determined by

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{v} = \mathbf{e}_1, \quad \lambda = \omega^2.$$  \hspace{1cm} (5.36)

By the Cramer’s rule we have
\[ v_1 = \frac{\det(\hat{K} - \lambda \hat{M})}{\det(K - \lambda M)} \] (5.37)

where \( v_1 \), the first element of \( v \), is called the collocated sine response function at \( x_1 \).

The important consequence of the above analysis is that \( \mu_1, \mu_2, \ldots, \mu_{n-1} \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the zeros and poles of \( v_1 \), respectively. The data needed for the construction may therefore determined by a simple vibration test, and there is no need to impose the restriction of no motion of \( m_1 \).

5.d. The Reconstruction

The discussion so far was motivated by the need of developing the eigenvalue-eigenvector relation (5.29). With this result at hand the reconstruction is described in the following manuscript (to appear in Encyclopedia of Vibration, Academic Press, London, 1999). Note that the eigenvalue-eigenvector relation (equation (20) in the manuscript) is indeed the key to solving the problem.

INVERSE PROBLEMS

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Many problems in physics, science and engineering science can be described schematically in the block diagram form shown in Figure 1.

![Figure 1](image)

**Figure 1** Input-output relation

For example, the vector differential equation governing the motion of a linear vibrating system is given by

\[ M \ddot{x} + C \dot{x} + K x = f(t), \] (1)

where \( M \), \( C \) and \( K \) are the mass, damping and stiffness matrices and \( f(t) \) is the vector of the external force. The block diagram takes for this case the form shown in Figure 2.
Problems where the system and the input are given, and the output is to be determined, are classified as direct problems. Inverse problems are those where the system is to be found based on the knowledge of the input and the output.

The algebraic eigenvalue problem plays an important role in vibration analysis. The eigenvalue problem associated with (1) is to determine the poles \( s_i \) and their associated modes \( v_i \neq 0 \) that satisfy

\[
(s^2 M + s C + K) v_i = 0.
\]  

Problems of reconstructing physical parameters appearing in the matrices \( \mathbf{M}, \mathbf{C}, \mathbf{K} \) based on partial knowledge of the poles and modes of the system are called inverse vibration problems.

The Classical Problem.

Perhaps the most celebrated inverse problem in vibration is that of reconstructing using spectral data a simply connected \( n \)-degree-of-freedom mass-spring system, such as that shown in Figure 3(a). The eigenvalue problem associated with this system may be written in the form

\[
\begin{bmatrix}
\mathbf{m}_1 & \mathbf{k}_1 & \mathbf{m}_2 & \mathbf{k}_2 & \cdots & \mathbf{m}_{n-2} & \mathbf{k}_{n-2} & \mathbf{m}_{n-1} & \mathbf{k}_{n-1} & \mathbf{m}_n
\end{bmatrix}
\]

(a)

(b)
\[(K - \lambda M)\mathbf{v} = \mathbf{0}, \quad \lambda = -s^2, \quad (3)\]

where \(K\) is a tridiagonal symmetric matrix

\[
K = \begin{bmatrix}
k_1 & -k_1 & & & \\
- k_1 & k_1 + k_2 & -k_2 & & \\
& -k_2 & k_2 + k_3 & -k_3 & \\
& & \ddots & \ddots & \ddots \\
& & & -k_{n-1} & k_{n-1} + k_n
\end{bmatrix}, \quad (4)
\]

and \(M\) is diagonal

\[
M = diag\{m_1, m_2, m_3, \ldots, m_n\}. \quad (5)
\]

This problem has \(n\) distinct positive eigenvalues, which may be ordered in the form

\[
\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n. \quad (6)
\]

If the mass \(m_1\) is attached to the ground via a rigid support, as shown in Figure 3(b), then the constrained system has \(n - 1\) degrees-of-freedom with eigenvalue problem

\[(\hat{K} - \mu \hat{M})\hat{v} = \mathbf{0}, \quad (7)\]

where \(\hat{K}\) is an \((n-1) \times (n-1)\) tridiagonal symmetric matrix determined by omitting the first row and column of \(K\),

\[
\hat{K} = \begin{bmatrix}
k_1 + k_2 & -k_2 & & & \\
- k_2 & k_2 + k_3 & -k_3 & & \\
& -k_3 & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & -k_{n-1} & k_{n-1} + k_n
\end{bmatrix}, \quad (8)
\]

and similarly \(\hat{M}\) is a diagonal matrix obtained by removing the first row and column of \(M\),

\[
\hat{M} = diag\{m_2, m_3, \ldots, m_n\}. \quad (9)
\]
The distinct real positive eigenvalues $\mu_i$ of this system may be numbered in an increasing order

$$\mu_1 < \mu_2 < \cdots < \mu_{n-1}. \quad (10)$$

Knowing the total mass

$$m_T = \sum_{i=1}^{n} m_i, \quad (11)$$

and the two sets of eigenvalues (6) and (10) the masses $m_i > 0$ and the springs $k_i > 0$, $i = 1, 2, \ldots, n$, can be determined uniquely, provided that the interlacing property

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \cdots < \mu_{n-1} < \lambda_n \quad (12)$$

holds.

In order to show how the physical parameters of the system may be determined from the given data a matrix $A$ is defined such that

$$A = M^{\frac{1}{2}} K M^{\frac{1}{2}}, \quad (13)$$

where

$$M^{\frac{1}{2}} = diag\left\{ \frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}} \right\}. \quad (14)$$

The matrix $A$ is symmetric tridiagonal

$$A = \begin{bmatrix}
\alpha_1 & \beta_1 & & \\
\beta_1 & \alpha_2 & \beta_2 & \\
& \beta_2 & \alpha_3 & \beta_3 & \\
& & \ddots & \ddots & \ddots & \\
& & & \beta_{n-1} & \alpha_n \\
\end{bmatrix}, \quad (15)$$

and being congruently equivalent to the matrix pencil $K - \lambda M$ it has the eigenvalues (6). Moreover, a matrix $\hat{A}$ which is formed by removing the first row and column of $A$, i.e.
\[
\hat{A} = \begin{bmatrix}
\alpha_2 & \beta_2 \\
\beta_2 & \alpha_3 & \beta_3 \\
\vdots & \ddots & \ddots & \ddots \\
& & \beta_{n-1} & \alpha_n
\end{bmatrix}, \quad (16)
\]

has the eigenvalues (10).

Denote the spectral decomposition of \(A\) by

\[
A V = V \Lambda, \quad (17)
\]

where \(V = [v_j]\) is orthogonal

\[
V^T V = I \quad (18)
\]

and

\[
\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}. \quad (19)
\]

Then the \(j\)-th element of the first row of \(V\) can be determined by the eigenvalue-eigenvector relation

\[
v_{ij}^2 = \frac{\prod_{i=1}^{n-1} (\lambda_j - \mu_i)}{\prod_{i=1}^{n} (\lambda_j - \lambda_i)}. \quad (20)
\]

The first row of (17) gives the following set of equations

\[
\begin{align*}
\alpha_1 v_{11} + \beta_1 v_{21} &= \lambda_1 v_{11} \\
\alpha_1 v_{12} + \beta_1 v_{22} &= \lambda_2 v_{12} \\
&\vdots \\
\alpha_1 v_{1n} + \beta_1 v_{2n} &= \lambda_n v_{1n}
\end{align*} \quad (21)
\]

by virtue of (15). Multiplying the \(i\)-th equation of (21) by \(v_{1i}\) yields

\[
\begin{align*}
\alpha_1 v_{11}^2 + \beta_1 v_{11} v_{21} &= \lambda_1 v_{11}^2 \\
\alpha_1 v_{12}^2 + \beta_1 v_{12} v_{22} &= \lambda_2 v_{12}^2 \\
&\vdots \\
\alpha_1 v_{1n}^2 + \beta_1 v_{1n} v_{2n} &= \lambda_n v_{1n}^2
\end{align*} \quad (22)
\]
Adding these equations gives

$$\alpha_1 = \sum_{i=1}^{n} \lambda_i v_i^2$$

(23)

since \(\sum_{i=1}^{n} v_i^2 = 1\) and \(\sum_{i=1}^{n} v_i v_{2i} = 0\) by virtue of the orthogonal relation (18). So \(\alpha_1\) is determined by the given data via (20) and (23).

It also follows from (21) that

\[
\begin{align*}
(\alpha_1 - \lambda_1)^2 v_{11}^2 &= \beta_1^2 v_{21}^2 \\
(\alpha_1 - \lambda_2)^2 v_{12}^2 &= \beta_1^2 v_{22}^2 \\
&\vdots \\
(\alpha_1 - \lambda_n)^2 v_{1n}^2 &= \beta_1^2 v_{2n}^2
\end{align*}
\]

(24)

Hence adding the equations in (24) and using the orthogonal relation \(\sum_{i=1}^{n} v_i^2 = 1\) gives

$$\beta_1 = -\sqrt{\sum_{i=1}^{n} (\alpha_1 - \lambda_i)^2 v_{1i}^2}$$

(25)

which determines \(\beta_1\).

Knowing \(\alpha_1\) and \(\beta_1\), the second row of the eigenvector matrix \(V\) can be obtained by (21). The other elements of \(A\), \(\alpha_j\) and \(\beta_j\), may then be determined successively for \(j = 2, 3, \ldots, n\) by following a similar process in which \(\alpha_j\) and \(\beta_j\) are determined from the set of equations

\[
\begin{align*}
\beta_{j-1}v_{j-1,1} + \alpha_j v_{j,1} + \beta_j v_{j+1,1} &= \lambda_1 v_{j,1} \\
\beta_{j-1}v_{j-1,2} + \alpha_j v_{j,2} + \beta_j v_{j+1,2} &= \lambda_2 v_{j,2} \\
&\vdots \\
\beta_{j-1}v_{j-1,n} + \alpha_j v_{j,n} + \beta_j v_{j+1,n} &= \lambda_n v_{j,n}
\end{align*}
\]

(26)

describing the elements of the \(j\)-th row of (17) with \(\beta_n = v_{n+1,i} = 0\). The interlacing property (12) ensures that the diagonal elements of \(A\) are all positive and that \(\beta_j < 0\) for \(j = 1, 2, \ldots, n - 1\).

After reconstructing \(A\) the mass and stiffness matrices can be evaluated as follows. Multiplying (13) by \(M^\frac{1}{2} p\), where
\[
p = \begin{pmatrix}
\frac{\sqrt{m_n}}{k_n} \\
1 \\
\vdots \\
1
\end{pmatrix},
\]

(27)

gives

\[
\frac{1}{2} AM^{\frac{1}{2}} p = M^{\frac{1}{2}} K p.
\]

(28)

Or, by virtue of (4), (14) and (27)

\[
Ay = e_n
\]

(29)

where

\[
y = \frac{1}{k_n} \begin{pmatrix}
\sqrt{m_1} \\
\sqrt{m_2} \\
\vdots \\
\sqrt{m_n}
\end{pmatrix}
\]

(30)

and \( e_n \) is the \( n \)-th unit vector \( e_n = (0 \cdots 0 1)^T \). Knowing \( A \) the vector \( y \) can be determined from (29). Then

\[
y^T y = \frac{m_p}{k_n} \sum_{i=1}^n m_i = \frac{m_n m_T}{k_n}
\]

(31)

implies that

\[
\frac{\sqrt{m_n}}{k_n} = \sqrt{\frac{y^T y}{m_T}},
\]

(32)

and hence the masses \( m_i, i = 1,2,\ldots,n \), are determined by (30) and (32). Knowing \( M \) and \( A \) the stiffness matrix is found by

\[
K = M^{\frac{1}{2}} A M^{\frac{1}{2}}
\]

(33)

which completes the reconstruction of the system.
Other Problems.

One other problem that has attracted much attention is the problem of reconstructing the physical parameters $m_i$, $k_i$, and $d_i$ of a mass-spring-rod system, such as that shown in Figure 4. The parameters are essentially constructed from three sets of spectral data, e.g., fixed-free, fixed-fixed and fixed-simply-supported configurations. This problem is associated with the reconstruction of a five-diagonal symmetric matrix. The conditions ensuring the reconstruction of a realizable system with positive parameters have been found by Gladwell and they are more involved than the simple interlacing properties.

![Figure 4 Mass-spring-rod system.](image)

There are several variations of the classical problem of reconstructing the mass-spring system from spectral data. Instead of fixing the left mass of Figure 1(a) a mass or spring may be attached to the free end. If a damper is attached to the free end, as shown in Figure 5, then $m_i$, $k_i$, and $c$ can be reconstructed apart from a scale factor by knowing only the $2n$ (complex) poles of the damped system. This interesting and result is due to K Veselic.

![Figure 5 Vecelic's system.](image)

Inverse mode problems are those where the reconstruction of the system is based on eigenvector and eigenvalue data. For example, the mass-spring system of Figure 1(a) may be reconstructed from two mode-shapes, one eigenvalue, and the total mass of the system. Similarly the mass-spring-rod system of Figure 4 may be reconstructed from three eigenvectors, two eigenvalues and the total mass.
The problems mentioned so far are all finite dimensional systems. In analogy the physical parameters of distributed parameter systems may be reconstructed from spectral and modal data. For example, the axial rigidity and density functions of a non-uniform axially vibrating rod are determined by two sets of spectral data associated with the fixed-free and fixed-fixed configurations. Alternatively two eigenfunctions, an eigenvalue and the total mass determine the rigidity and density of the rod. Many of the inverse problems associated with finite dimensional systems may be regarded as a certain discrete version of a distributed parameter problem. It should be noted however that since the asymptotic behavior of eigenvalues of a distributed parameter system is different from that of its related discrete model, there is no reliable way to reconstruct the physical parameters of a distributed parameter system by considering a related discrete system.

Concluding Remarks

The problem of reconstructing the mass-spring system from two spectral sets and the total mass studied above is fully solved in the following sense:

• there exists a unique realizable system, with positive masses and springs, that fits the prescribed data whenever the interlacing property (12) holds,
• there is no realizable system when the spectral data violate (12), and
• direct methods for reconstructing the system do exist.

Only a few other inverse problems in vibration have been posed and fully solved in this sense. For many other inverse problems the spectral data do not allow the unique reconstruction of a realistic system. For some problems there exists a continuous family of solutions. In other problems there are finite number of systems satisfying the given data. The necessary and sufficient conditions, which allow the construction of a realistic system, are not always known. For some problems there is no direct method of solution and iterative methods are used instead. Many iterative methods do not always converge.

This suggests that the subject is still in its development phase. It started with Gantmakher and Krein in 1950 with a problem similar to reconstructing the mass-spring system from spectral data. Highlighted by Gladwell in his monograph Inverse Problems in Vibration describing the state of the art of knowledge, including his own major contribution in the field. It is most probably that related research will continue to be carried out in the future developing the subject further.

Further Reading