HARMONIC EXCITATION OF A SINGLE DEGREE OF FREEDOM SYSTEM.

2.1 Harmonic Excitation:

The differential equation of motion, for the system shown below, is given by

\[ m\ddot{x} + c\dot{x} + kx = f(t) \]  \hspace{1cm} (2.1.1)

where \( f(t) \) is the applied force.

If

\[ f(t) = F_0 \sin \omega t, \]  \hspace{1cm} (2.1.2)

where \( F_0 \) is a constant, then the system is said to be a harmonically excited system. The differential equation of motion for a harmonically excited system is thus

\[ m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t. \]  \hspace{1cm} (2.1.3)

We now wish to find the general solution of (2.1.3). Then the response of the system can be determined by the initial conditions \( x(0) \) and \( \dot{x}(0) \).
Let \( x_H \) be the \textit{general} solution of the homogeneous problem
\[
m \ddot{x}_H + c \dot{x}_H + kx_H = 0, \tag{2.1.4}
\]
and let \( x_P \) be a particular solution of
\[
m \ddot{x}_P + c \dot{x}_P + kx_P = F_0 \sin \omega t. \tag{2.1.5}
\]

Then

\textbf{Theorem 2.1.1}

\[
x(t) = x_H(t) + x_P(t) \tag{2.1.6}
\]

is the general solution of (3).

\textbf{Proof.}

Since \( x_H(t) \) is the general solution of (4), it is expressed in terms of a linear combination of two linearly independent functions. Hence, by (6), \( x(t) \) is also expressed in terms of a linear combination of two linearly independent functions. Thus, all we need to show is that \( x(t) \) in (6) satisfies the differential equation (3).

By (6)
\[
\dot{x} = \dot{x}_H + \dot{x}_P \tag{2.1.7}
\]
and
\[
\ddot{x} = \ddot{x}_H + \ddot{x}_P. \tag{2.1.8}
\]

Using (6)-(8) we have
\[
m \dddot{x} + c \ddot{x} + kx \\
= m(\dddot{x}_H + \dddot{x}_P) + c(\ddot{x}_H + \ddot{x}_P) + k(\dot{x}_H + \dot{x}_P) \tag{2.1.9}
\]
\[
= (m \dddot{x}_H + c \ddot{x}_H + kx_H) + (m \dddot{x}_P + c \ddot{x}_P + kx_P)
\]
and by (4) and (5)
\[
(m \dddot{x}_H + c \ddot{x}_H + kx_H) + (m \dddot{x}_P + c \ddot{x}_P + kx_P)
\]
\[
= 0 + F_0 \sin \omega t. \tag{2.1.10}
\]

It thus follows from (9) and (10) that \( x(t) \), given by (6), satisfies (3). \( \Box \)
The general solution $x_H(t)$, of the homogeneous problem (4) has been derived in section 1.5. It is given by (see (1.5.6))

$$x_H(t) = \begin{cases} 
A_1 e^{-\zeta \omega_n t} \sin \omega_d t + B_1 e^{-\zeta \omega_n t} \cos \omega_d t; & \zeta < 1 \\
A_2 e^{-\omega_n t} + B_2 t e^{-\omega_n t}; & \zeta = 1. \\
A_3 e^{(-\zeta + \sqrt{\zeta^2 - 1}) \omega_n t} + B_3 e^{(-\zeta - \sqrt{\zeta^2 - 1}) \omega_n t}; & \zeta > 1 
\end{cases} \quad (2.1.11)$$

We now show how to determine a particular solution $x_p$ of (5). We try a solution of the form

$$x_p(t) = X \sin(\omega t - \phi), \quad (2.1.12)$$

where $X$ and $\phi$ are constants and, as in (3), $\omega$ is the frequency of the exciting force. Substituting (12) into (5) gives

$$-m \omega^2 X \sin(\omega t - \phi) + c \omega X \cos(\omega t - \phi) + kX \sin(\omega t - \phi) = F_0 \sin \omega t, \quad (2.1.13)$$

or, using the identity $\cos(\alpha) = \sin(\alpha + \frac{\pi}{2})$, we have

$$-m \omega^2 X \sin(\omega t - \phi) + c \omega X \sin(\omega t - \phi + \frac{\pi}{2}) + kX \sin(\omega t - \phi) = F_0 \sin \omega t. \quad (2.1.14)$$

This is a vector equation with the graphical interpretation shown below.
This polygon gives
\[ F_0^2 = (kX - m\omega^2 X)^2 + (c\omega X)^2, \]  
(2.1.15)
or
\[ X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}. \]  
(2.1.16)

We also have
\[ \tan \phi = \frac{c\omega X}{kX - m\omega^2 X} = \frac{c\omega}{k - m\omega^2}, \]  
(2.1.17)
or
\[ \phi = \tan^{-1} \frac{c\omega}{k - m\omega^2}. \]  
(2.1.18)

In summary, a particular solution \( x_p \) of (5) is given by (12), with \( X \) and \( \phi \) as in (16) and (18), respectively.

**Ex. 2.1.1:** (a) Show that, say, \( X\sin(\omega t - \phi) + 1.23456e^{-\zeta \omega_n t} \sin(\omega_d t) \), is another particular solution which solves (5).

(b) Can you find another particular solution of (5) ?

By Theorem 2.1.1 the general solution of (3) is

\[
x(t) = \begin{cases}
  A_1e^{-\zeta \omega_n t} \sin \omega_d t + B_1e^{-\zeta \omega_n t} \cos \omega_d t \\
  + X \sin(\omega t - \phi); & \zeta < 1 \\
  A_2e^{-\omega_n t} + B_2te^{-\omega_n t} + X \sin(\omega t - \phi); & \zeta = 1 \\
  A_3e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + B_3e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \\
  + X \sin(\omega t - \phi); & \zeta > 1
\end{cases}
\]  
(2.1.19)
where $X$ and $\phi$ are given by (16) and (18), respectively. Alternatively, using (1.5.2) and (1.5.3), we may write $X$ and $\phi$ in terms of $\omega_n$ and $\zeta$, as

$$X = \frac{F_0}{k} \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

and

$$\phi = \tan^{-1} \left( \frac{2\zeta \omega / \omega_n}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right).$$

**Ex. 2.1.2:** Show that (16) and (18) imply (20) and (21), respectively.

We see that if $c=0$ and the exciting frequency $\omega = \omega_n = \sqrt{\frac{k}{m}}$ then the amplitude $X \to \infty$. This is the *resonance* phenomenon. To avoid resonance we should introduce damping and avoid harmonic excitation with frequency equal to the natural frequency of the system.

The general solution (19) shows that for $\zeta>0$ the homogeneous part of the solution vanishes as $t \to \infty$. We therefore say that the particular solution $x_P$, given by (12), is the *steady state response* of the system.
Example 2.1.1

We have mentioned (see sections 1.4 and 1.6) that

\[ x(t) = A \sin(\omega t) + B \cos(\omega t) \]  

(2.1.22)
can be expressed equivalently in the form

\[ x(t) = C \sin(\omega t - \alpha) \].  

(2.1.23)

Determine the explicit relations between the constants \( A, B \) and the constants \( C, \alpha \) by using the graphical method described in this section.

**Solution.**

Equating (22) and (23) we have

\[ C \sin(\omega t - \alpha) = A \sin(\omega t) + B \cos(\omega t) \]  

(2.1.24)
or

\[ C \sin(\omega t - \alpha) = A \sin(\omega t) + B \sin(\omega t + \frac{\pi}{2}), \]  

(2.1.25)

which has the graphical interpretation shown below.

![Graphical Interpretation](image)

We therefore have

\[ C = \sqrt{A^2 + B^2} \]  

(2.1.26)

and

\[ \alpha = -\tan^{-1} \frac{B}{A}. \]  

(2.1.27)
Alternatively, we may arrive at these expressions in the following way. Using the trigonometric identity
\[ \sin(\beta + \gamma) = \sin \beta \cos \gamma + \cos \beta \sin \gamma \] (2.1.28)
we have
\[ C \sin(\omega t - \alpha) = C \cos(-\alpha) \sin(\omega t) + C \sin(-\alpha) \cos(\omega t) = C \cos \alpha \sin(\omega t) - C \sin \alpha \cos(\omega t). \] (2.1.29)
Thus, comparing (29) and (22) yields
\[ A = C \cos \alpha \] (2.1.30)
and
\[ B = -C \sin \alpha. \] (2.1.31)
Adding the squares of (30) and (31) gives
\[ A^2 + B^2 = C^2 \cos^2 \alpha + C^2 \sin^2 \alpha = C^2, \] (2.1.32)
which implies (26).
Dividing (31) by (30) gives
\[ -\tan \alpha = \frac{B}{A}, \] (2.1.33)
which implies (27).

Q. Which of the above methods appears to be more direct and simpler to apply?
Example 2.1.2: Rotating unbalanced.

Consider the system of total mass $M$, with the rotating mass $W$ shown below. The mass $W$ rotates about $o$ with constant angular velocity $\omega$ and radius of rotation $e$. The system is supported by a spring of constant $k$ and a damper of constant $c$. Determine the amplitude of vibration of $M$ at steady state.

Solution:

Without lost of generality we may assume that the mass of the non-rotating structure is lumped at $o$ (Why this is so?). Then we have

Position of $M-W \rightarrow x$.

Position of $W \rightarrow x + e \sin(\omega t)$.

The differential equation of motion is, therefore,

$$ (M - W)\ddot{x} + W \frac{d^2}{dt^2} [x + e \sin(\omega t)] + c\dot{x} + kx = 0 $$  \hspace{1cm} (2.1.34)

or

$$ M\ddot{x} - W\ddot{x} + W\ddot{\omega}^2 e \sin(\omega t) + c\dot{x} + kx $$

$$ M\ddot{x} - W\omega^2 e \sin(\omega t) + c\dot{x} + kx = 0 $$ \hspace{1cm} (2.1.35)

The differential equation of motion can be written in the form

$$ M\ddot{x} + c\dot{x} + kx = W\omega^2 e \sin(\omega t) $$ \hspace{1cm} (2.1.36)
Note that $W\omega^2 e$ is constant. Hence substituting $W\omega^2 e$ for $F_0$ and $M$ for $m$ in (16), we find that the steady state amplitude of vibration is given by

$$X = \frac{W\omega^2 e}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}.$$  \hfill (2.1.37)
2.2 Harmonic Base Excitation.

The base of the system shown below moves according to the function

\[ y(t) = Y \sin(\omega t), \quad (2.2.1) \]

where \( Y \) is the constant amplitude of the base motion.

Let \( x(t) \) be the displacement of the mass \( m \) from its static position. Then, the spring force is \( k(x - y) \), the damper force is \( c(\dot{x} - \dot{y}) \), and the differential equation of motion is

\[ m\ddot{x} + c(\ddot{x} - \ddot{y}) + k(x - y) = 0. \quad (2.2.2) \]

Denote

\[ z = x - y. \quad (2.2.3) \]

Then (2) gives

\[ m \ddot{z} + c \dot{z} + kz = -m\ddot{y}. \quad (2.2.4) \]

From (1) we have

\[ \ddot{y} = -Y \omega^2 \sin(\omega t). \quad (2.2.5) \]

Hence, substituting (5) in (4) yields

\[ m \ddot{z} + c \dot{z} + kz = mY \omega^2 \sin(\omega t), \quad (2.2.6) \]

which is a differential equation of motion similar to (2.1.3), with

\[ F_0 = mY \omega^2. \]

By (2.1.16) the steady state amplitude for this case is

\[ X = \frac{mY \omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad (2.2.7) \]
and the phase lag between the motion of the base and the mass is given by (2.1.18).
2.3 Transmissibility of Vibration.

One important aspect of vibration analysis in application, is determination of the steady state amplitude of vibration under harmonic excitation. The other important factor is determination of the force $f_T(t)$ transmitted at steady state from the vibrating system to its supporting structure. We now address the later issue.

The force transmitted from the mass-spring-system, shown below, to the ground consists of two components, the damper and the spring forces, ie.

$$f_T(t) = c\dot{x} + kx.$$  \hspace{1cm} (2.3.1)

It has been shown in section 2.1 that at steady state the system vibrates with harmonic motion

$$x(t) = X \sin(\omega t - \phi),$$  \hspace{1cm} (2.3.2)

and that the following equation is satisfied (see (2.1.14))

$$-m\omega^2 X \sin(\omega t - \phi) + c\omega X \sin(\omega t - \phi + \frac{\pi}{2}) + kX \sin(\omega t - \phi) = F_0 \sin\omega t.$$  \hspace{1cm} (2.3.3)

Hence, by (1) we have

$$f_T = c\omega X \sin(\omega t - \phi + \frac{\pi}{2}) + kX \sin(\omega t - \phi).$$  \hspace{1cm} (2.3.4)

The force transmitted to the ground is, therefore, harmonic. Let

$$f_T(t) = F_T \sin(\omega t - \alpha).$$  \hspace{1cm} (2.3.5)
where $F_T$ is constant. Then equations (3)-(5) have the graphical interpretation shown below.

It follows from this polygon that

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = X\sqrt{k^2 + (c\omega)^2}$$  \hspace{0.5cm} (2.3.6)

and

$$\alpha = \phi - \beta = \tan^{-1}\frac{c\omega}{k - m\omega^2} - \tan^{-1}\frac{c\omega}{k}.$$  \hspace{0.5cm} (2.3.7)

Define the transmissibility $T_R$ as the ratio

$$T_R = \frac{F_T}{F_0}.$$  \hspace{0.5cm} (2.3.8)

Then by (2.1.15) and (6) we have

$$T_R = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}.$$  \hspace{0.5cm} (2.3.9)

We may write this equation in terms of $\omega$ and $\zeta$ as

$$T_R = \sqrt{\frac{1 + (2\zeta\frac{\omega}{\omega_n})^2}{[1 - (\frac{\omega}{\omega_n})^2]^2 + (2\zeta\frac{\omega}{\omega_n})^2}}.$$  \hspace{0.5cm} (2.3.10)
It follows from (9) that increasing $m$, such that $(k - m\omega^2)^2$ becomes dominant, reduces the transmissibility $T_R$, since $m$ appears only in the denominator. The problem of reducing $T_R$ while keeping $\omega_n$ constant, is an interesting optimisation problem (see (10)).