5. VIBRATIONS OF CONTINUOUS SYSTEMS

5.1 The vibrating string.

Consider a uniform string, of mass per unit length \( \rho \), stretched under tension \( T \), between two supports distance \( L \) apart, as shown below.

With the assumption of small vibrations about equilibrium, the tension \( T \) in the spring may be considered constant, ie. independent of the time \( t \) and the position \( x \). The free body diagram for the string’s element of infinitesimal length \( dx \), laid originally in the interval \([x, x+dx]\), is shown below.

Thus, the mass of the element is \( \rho dx \), its acceleration is \( \frac{\partial^2 u}{\partial t^2} \), and the Newton’s second law, applied in the \( u \) direction, gives

\[
\rho \ dx \frac{\partial^2 u}{\partial t^2} = T \left( \theta + \frac{\partial \theta}{\partial x} \ dx \right) - T \theta ,
\]

which simplifies to

\[
\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial \theta}{\partial x} .
\]
But, by definition the slop \( \theta(x) \) is

\[
\theta = \frac{\partial u}{\partial x}.
\]  

(5.1.3)

Hence, substituting (3) in (2) gives

\[
\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2},
\]  

(5.1.4)

or

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]  

(5.1.5)

where

\[
c = \sqrt{\frac{T}{\rho}}.
\]  

(5.1.6)

The parameter \( c \) is known as the *speed propagation of waves* in the string. The dynamic of the string is determined by the differential equation (5), the boundary conditions

\[
\begin{align*}
u(0,t) &= 0 \\
u(L,t) &= 0
\end{align*}
\]  

(5.1.7)

and the initial conditions

\[
\begin{align*}
u(x,0) &= f(x) \\
\frac{\partial u}{\partial t}(x,0) &= g(x)
\end{align*}
\]  

(5.1.8)

where \( f(x) \) and \( g(x) \) describe the displacement and velocity of the string at the time \( t=0 \), respectively. Equations (5)-(7) characterise the string and its configuration. The properties of the system, ie, its natural frequencies and mode shapes of vibrations, are thus determined by (5)-(7) alone. To obtain the
actual displacement \( u(x,t) \) of (5), for each time \( t \) and point \( x \), we must use the initial conditions (8) as well.

Let us now determine the natural frequencies and mode shapes of the string. As done in the previous chapters, we assume that the string vibrates with harmonic motion of the form

\[
\begin{align*}
  u(x,t) &= v(x) \sin \omega t . & (5.1.9)
\end{align*}
\]

Note that by this assumption we have separated the variables \( x \) and \( t \).

Substituting (9) in (5) gives

\[
-\omega^2 v \sin \omega t = c^2 v'' \sin \omega t ,
\]

or

\[
v'' + \left( \frac{\omega}{c} \right)^2 v = 0 ,
\]

where primes denotes derivatives with respect to \( x \). Substituting (9) in (7) gives

\[
\begin{cases}
  v(0) \sin \omega t = 0 \\
  v(L) \sin \omega t = 0 .
\end{cases}
\]

An assumption that \( \sin \omega t = 0 \) leads through (9) to a trivial solution \( u(x,t) = 0 \).

Such a solution is not of interest since it predicts the dynamics of the string only in the case where the initial conditions (8) vanish. Trivially there is no motion at all in this case. We thus conclude that (12) implies

\[
\begin{cases}
  v(0) = 0 \\
  v(L) = 0 .
\end{cases}
\]

Combining (11) and (13) we obtain the eigenvalue problem

\[
\begin{cases}
  v'' + \lambda v = 0 \\
  v(0) = 0, \quad v(L) = 0 ,
\end{cases}
\]

where
\[
\lambda = \left( \frac{\omega}{c} \right)^2.
\] (5.1.15)

In analogy with the discrete eigenvalue problem \((\mathbf{K} - \lambda \mathbf{M}) \phi = \mathbf{0}\), the continuous eigenvalue problem (14) requires evaluation of the eigenvalues \(\lambda_i\) and their associated eigenfunctions \(v_i(x)\). The general solution of the differential equation in (14) is

\[
v(x) = A_1 \sin \sqrt{\lambda} x + A_2 \cos \sqrt{\lambda} x, \quad (5.1.16)
\]

where \(A_1\) and \(A_2\) are arbitrary constants.

**Proof.**

It follows from (16) that

\[
v'' = -A_1 \lambda \sin \sqrt{\lambda} x - A_2 \lambda \cos \sqrt{\lambda} x.
\]

Hence the differential equation in (14) gives

\[
v'' + \lambda v
\]
\[
= -A_1 \lambda \sin \sqrt{\lambda} x - A_2 \lambda \cos \sqrt{\lambda} x + \lambda (A_1 \sin \sqrt{\lambda} x + A_2 \cos \sqrt{\lambda} x),
\]
\[
= 0
\]

and \(v(x)\) is expressed in terms of two arbitrary constants. \(\square\)

The left boundary condition in (14) yields

\[
v(0) = A_2 = 0, \quad (5.1.17)
\]

so that (16) reduces to

\[
v(x) = A_1 \sin \sqrt{\lambda} x. \quad (5.1.18)
\]

The right boundary condition in (14) gives

\[
v(L) = A_1 \sin \sqrt{\lambda} L = 0. \quad (5.1.19)
\]
Selecting $A_1=0$ leads to the trivial solution $v(x)=u(x,t)=0$. We therefore choose

$$\sin \sqrt{\lambda} L = 0, \quad (5.1.20)$$

and obtain the eigenvalues $\lambda_i$

$$\sqrt{\lambda_i} L = i\pi, \quad i = 1, 2, 3, \ldots \quad (5.1.21)$$

The natural frequencies $(\omega_n)_i$ of the string are determined by using (15)

$$\left(\omega_n\right)_i = \frac{ic\pi}{L}, \quad i = 1, 2, 3, \ldots \quad (5.1.22)$$

**Natural frequencies of a fixed-fixed string**

By (18) their associated eigenfunctions $v_i$ are

$$v_i(x) = \sin \frac{i\pi}{L} x \quad (5.1.23)$$

**Eigenfunctions, or mode-shapes, of the fixed-fixed string**

In the discrete case the eigenvectors can be scaled arbitrarily. Similarly, the eigenfunctions of the continuous system are determined up to a scalar constant $A_j$ (see (19)). In (23) we use (19) with $A_j=1$.

We know that resonance occurs when a multi-degree-of-freedom system is excited by a harmonic force with frequency that is equal to a natural frequency. Similarly, the string vibrates in resonance when a harmonic force $F_0 \sin \omega_n t$ is applied to the string, where $F_0$ is some constant and $\omega_n$ is one of the natural frequencies (22) of the string.

We now give a physical interpretation to the mode shapes (or eigenfunction). If the initial conditions are
\[ \begin{align*}
  u(x,0) &= v_i(x) \\
  \frac{\partial u}{\partial t}(x,0) &= 0
\end{align*} \]  

(5.1.24)

where \( v_i \) is the \( i \)-th mode shape of the system, then each point of the string vibrates in a single harmonic frequency \( \left( \omega_n \right)_i \). More precisely the string vibrates in this case according to

\[ u(x,t) = v_i(x) \sin \frac{ic\pi}{L} t. \]  

(5.1.25)

**Proof.**

It is easily shown by direct substitution that

\[ u(x,t) = \sin \frac{i\pi x}{L} \sin \frac{ic\pi t}{L} \]

satisfies the differential equation (5), the boundary conditions (7) and the initial conditions (8) with

\[ f(x) = \sin \frac{i\pi x}{L} = v_i \]

and

\[ g(x) = 0. \]  

\[ \square \]
Consider the axially vibrating rod, of density $\rho(x)$, cross-sectional area $A(x)$, fixed at $x=0$ and free to oscillate at $x=L$, as shown below. Denote by $u(x,t)$ the displacement of the small element, laying originally in the interval $[x, x+dx]$. The free body diagram, for an instant where $0<u(x,t)<u(x+dx,t)$, is also shown in the figure below.

Noting that the mass of the element is $\rho A dx$, its acceleration $\frac{\partial^2 u}{\partial t^2}$, we find that the Newton’s second law gives

$$-P + \left( P + \frac{\partial P}{\partial x} \right) dx = \rho A dx \frac{\partial^2 u}{\partial t^2}, \quad (5.2.1)$$

or

$$\frac{\partial P}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2}. \quad (5.2.2)$$

Recall the stress-strain relation
\[ \sigma = E \varepsilon, \]  \hspace{1cm} (5.2.3) 

where \( \sigma \) in the applied stress, \( E \) is the *modulus of elasticity* and \( \varepsilon \) is the resulting strain. By definition

\[ \sigma = \frac{P}{A}, \]  \hspace{1cm} (5.2.4) 

and

\[ \varepsilon = \frac{\partial u}{\partial x}. \]  \hspace{1cm} (5.2.5) 

Hence (3) gives

\[ P = EA \frac{\partial u}{\partial x}. \]  \hspace{1cm} (5.2.6) 

Substituting (6) in (2) we obtain the differential equation of motion for the non-uniform rod

\[ \frac{\partial}{\partial x} \left( E(x) A(x) \frac{\partial u}{\partial x} \right) = \rho(x) A(x) \frac{\partial^2 u}{\partial t^2}. \]  \hspace{1cm} (5.2.7) 

*Equation of motion for a non-uniform axially vibrating rod*

For a uniform rod, \( A(x) = A, \ E(x) = E, \ \rho(x) = \rho \), and (7) is simplified to

\[ \frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}. \]  \hspace{1cm} (5.2.8) 

*Equation of motion for a uniform axially vibrating rod*

We denote the speed propagation in the uniform rod by

\[ c = \sqrt{\frac{E}{\rho}}, \]  \hspace{1cm} (5.2.9)
and write (8) in the form
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]  
(5.2.10)

identical to the equation governing the motion of the string (5.1.5). Note however, that for the rod \(c\) is given by (9), whereas for the string \(c\) is given by (5.1.6).

The boundary conditions for the fixed-free configuration of the rod imply that there is no displacement at \(x=0\), and no axial force at \(x=L\) (since there is no physical contact from the right at the free end). In view of (6) \(P(L)=0\) when the derivative of \(u\) with respect to \(x\) vanishes. The boundary conditions are, therefore,

\[
\begin{aligned}
  u(0,t) &= 0 \\
  \frac{\partial u(L,t)}{\partial x} &= 0 .
\end{aligned}
\]  
(5.2.11)

To determine the eigenvalue problem associated with the differential equation (10) and the boundary conditions (11) we assume harmonic motion of the form

\[
  u(x,t) = v(x) \sin \omega t
\]  
(5.2.12)

and, upon substitution of (12) in (10) and (11), obtain

\[
\begin{aligned}
  v'' + \lambda v &= 0 , \quad \lambda = \frac{\omega^2}{c^2} \\
  v(0) &= 0 \\
  v'(L) &= 0 .
\end{aligned}
\]  
(5.2.13)

The general solution (5.1.16) with the left boundary condition yields

\[
v = A_1 \sin \sqrt{\lambda} x .
\]  
(5.2.14)

The right boundary condition gives
\[ v'(L) = A_1 \sqrt{\lambda} \cos \sqrt{\lambda} L = 0. \quad (5.2.15) \]

Since \( A_1 = 0 \) or \( \lambda = 0 \) lead to the unwanted trivial solution \( u(x,t) = 0 \), we determine the frequency equation

\[ \cos \sqrt{\lambda} L = 0, \quad (5.2.16) \]

with the roots

\[ \sqrt{\lambda_i} L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \quad (5.2.17) \]

Using (13) we determine the natural frequencies of the fixed-free uniform rod

\[
\left( \omega_n \right)_i = \frac{(2i - 1)c\pi}{2L}, \quad i = 1,2,3,\ldots
\]

\textit{Natural frequencies of a fixed-free uniform rod} (5.2.18)

where \( c \) is given by (9).

Choosing \( A_1 = 1 \) arbitrarily the eigenfunction (14) are

\[
v_i = \sin \left( \frac{(2i - 1)\pi x}{2L} \right), \quad i = 1,2,3,\ldots
\]

\textit{Mode shapes of a fixed-free uniform rod} (5.2.19)

We could use other factor for \( A_1 \) and obtain another factor of the eigenfunction \( v_i \).
Example 5.1.1

Determine the frequency equation and mode shapes of the uniform axially vibrating rod of cross sectional area $A$, modulus of elasticity $E$ and density $\rho$, fixed at $x=0$ and spring supported at $x=L$, as shown below.

**Solution.**

The rod at static equilibrium, and in motion, is shown below.

This illustration shows that the right boundary condition is

$$\sigma(L,t) = -\frac{ku(L,t)}{A}, \quad (5.2.20)$$

where the minus sign in (20) is due to the fact that compressive stress is negative. By (3) and (5) we have

$$\sigma(L,t) = E \frac{\partial u(L,t)}{\partial x}, \quad (5.2.21)$$
so that the right boundary condition is

\[ E \frac{\partial u(L,t)}{\partial x} = -\frac{k u(L,t)}{A}. \]  

(5.2.22)

It follows that this rod is characterised by the differential equation (8)

\[ \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2}, \]  

(5.2.23)

with the boundary conditions

\[ \begin{align*}
    u(0,t) &= 0, \\
    \frac{\partial u(L,t)}{\partial x} &= -\frac{k}{EA} u(L,t). 
\end{align*} \]  

(5.2.24)

The harmonic motion assumption (12)

\[ u(x,t) = v(x) \sin \omega t, \]  

(5.2.25)

through an identical analysis to that done for the uniform fixed-free rod, leads to

\[ v(x) = A_1 \sin \frac{\sqrt{\rho} \omega x}{\sqrt{E}}, \]  

(5.2.26)

where \( v(x) \) is a mode shape and \( \omega \) is a natural frequency. Substituting (25) and (26) in the right boundary condition of (24) gives

\[ \frac{\sqrt{\rho} \omega}{\sqrt{E}} A_1 \cos \frac{\sqrt{\rho} \omega L}{\sqrt{E}} \sin \omega t = -\frac{k}{EA} A_1 \sin \frac{\sqrt{\rho} \omega L}{\sqrt{E}} \sin \omega t, \]  

(5.2.27)

or

\[ \tan \frac{\sqrt{\rho} L \omega}{\sqrt{E}} = -\frac{\sqrt{\rho} E A \omega}{k}. \]  

(5.2.28)

Frequency equation for the rod

By (26) the mode shapes for the rod are

\[ v(x) = \sin \frac{\sqrt{\rho} \omega x}{\sqrt{E}}, \]  

(5.2.29)

where \( \omega \) is any root of the frequency equation (28).
Denote by \( z \) the left and right side of (28). Then the natural frequencies \( \omega_i \) of the rod are the intersections of \( z = -\frac{\sqrt{\rho E A}}{k} \omega \) and \( z = \tan \frac{L \omega}{c} \), as shown below. We see from this graph that there is, as expected, infinite number of natural frequencies for the continuous rod. Note also that the fixed-free rod is a special case where \( k \to 0 \). The fixed-fixed configuration is another extreme, where \( k \to \infty \). It also follows from the graph below that the effect of increasing the spring constant \( k \) is to increase all the natural frequencies of the system. It was observed by Lord Rayleigh\(^*\) that this phenomenon applies not only for the vibrating rod but to all linear vibratory systems, discrete or continuous.

\[ z \]

\[ z = \tan \left( \frac{L \omega}{c} \right) \]

\[ \frac{c\pi}{2L} \quad \frac{c\pi}{L} \quad \frac{3c\pi}{2L} \quad \frac{2c\pi}{L} \quad \frac{5c\pi}{2L} \quad \frac{3c\pi}{L} \]

\( \omega_1 \quad \omega_2 \quad \omega_3 \)

\[ z = -\frac{\sqrt{\rho E A}}{k} \omega \]

- Natural frequencies of the fixed-spring supported rod
- Natural frequencies of the fixed-free rod
- Natural frequencies of the fixed-fixed rod

5.3 Approximating natural frequencies and mode shapes of a non-uniform rod by finite differences.

Consider the axially vibrating non-uniform rod of cross-sectional area $A(x)$, modulus of elasticity $E(x)$ and density $\rho(x)$, fixed at $x=0$ and free to vibrate at $x=L$, as shown below.

We found in the section 5.2 that this rod is characterised by the differential equation of motion (5.2.7)

$$\frac{\partial}{\partial x} \left( E(x) A(x) \frac{\partial u}{\partial x} \right) = \rho(x) A(x) \frac{\partial^2 u}{\partial t^2}$$

and the boundary conditions

$$\begin{cases} u(0,t) = 0 \\ \frac{\partial u(L,t)}{\partial x} = 0 \end{cases}$$

Harmonic motion assumption

$$u(x,t) = v(x) \sin \omega t,$$

leads to the associated eigenvalue problem

$$\begin{cases} (E(x) A(x) v'(x))' + \lambda \rho(x) A(x) v(x) = 0, \quad \lambda = \omega^2 \\ v(0) = 0 \\ v'(L) = 0 \end{cases}$$

where, as usual, primes denote derivatives with respect to $x$. 
Depending on the functions $A(x)$, $E(x)$ and $\rho(x)$, the eigenvalue problem (4) may not be solvable in closed form. We may approximate some eigenvalues and eigenvectors of (4) by using finite differences as follows.

Partition the rod into $n$ elements of equal length $h = L/n$ and denote the displacements of the right and left boundaries of the $i$-th element by $v_{i-1}$ and $v_i$, respectively, as shown below. Let $(EA)_i$ and $(\rho A)_i$ be the flexural rigidity and linear density of the mid-section of the $i$-th element.

Then a finite difference approximation for the $i$-th element gives

$$
\left( (EA') \right)_{x=(i-0.5)h} = \left( (EA)_i \frac{v_i - v_{i-1}}{h} \right)_{x=ix}, \quad (5.3.5)
$$

and similarly

$$
\left( (EA)_i \frac{v_i - v_{i-1}}{h} \right)_{x=ix} = \frac{(EA)_{i+1}}{h} \frac{v_{i+1} - v_i}{h} - \frac{(EA)_i}{h} \frac{v_i - v_{i-1}}{h}.
$$

The finite difference scheme for the differential equation in (4) is therefore
\[
\frac{(EA)_i}{h^2} v_{i-1} - \frac{(EA)_i + (AE)_{i+1}}{h^2} v_i + \frac{(EA)_{i+1}}{h^2} v_{i+1} + \lambda (pA)_i v_i = 0,
\]

or upon multiplying by \(-h\)

\[
-\frac{(EA)_i}{h} v_{i-1} + \frac{(EA)_i + (AE)_{i+1}}{h} v_i - \frac{(EA)_{i+1}}{h} v_{i+1} - \lambda h (pA)_i v_i = 0.
\]

Define

\[
k_i = \frac{(EA)_i}{h} \tag{5.3.9}
\]

and

\[
m_i = h(pA)_i. \tag{5.3.10}
\]

Then (8) can be written as

\[
-k_i v_{i-1} + (k_i + k_{i+1}) v_i - k_{i+1} v_{i+1} - \lambda mv_i = 0. \tag{5.3.11}
\]

In terms of finite differences, the left boundary condition in (4) gives

\[
v_0 = 0 \tag{5.3.12}
\]

and the right boundary implies that

\[
\frac{v_{n+1} - v_n}{h} = 0, \tag{5.3.13}
\]

or

\[
v_{n+1} = v_n. \tag{5.3.14}
\]

With (12), equation (11) gives for the first element, \(i=1\)
\[(k_1 + k_2)v_1 - k_2v_2 - \lambda mv_1 = 0. \quad (5.3.15)\]

With (14), equation (11) gives for the last element, \(i=n\)

\[-k_n v_{n-1} + (k_n + k_{n+1})v_n - k_{n+1}v_n - \lambda mv_n = -k_n v_{n-1} + k_nv_n - \lambda mv_n = 0. \quad (5.3.16)\]

Equation (11) for \(i=2,3,...,n-1\), together with (15) and (16) can be written in matrix form

\[(K - \lambda M)v = 0, \quad (5.3.17)\]

where

\[
K = \begin{bmatrix}
k_1 + k_2 & -k_2 & & \\
-k_2 & k_2 + k_3 & -k_3 & \\
& -k_3 & k_3 + k_4 & -k_4 \\
& & & & \ddots \\
& & & & -k_n & k_n
\end{bmatrix}, \quad (5.3.18)
\]

\[
M = \begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
& \ddots \\
m_n
\end{bmatrix}, \quad (5.3.19)
\]

and

\[v = (v_1, v_2, \ldots, v_n)^T. \quad (5.3.20)\]

Equation (17) is the well known generalised eigenvalue problem. It can be solved by standard routines (e.g. \texttt{eig(K,M)} in Matlab). The eigenvalues of (17) approximate the square of the natural frequencies of the non-uniform rod. The eigenvectors of (17) are the approximation of the corresponding mode shapes, measured at \(x=ih, i=1,2,\ldots,n\).
This result has an interesting interpretation. We note that $m_i$ in (10) is the mass of the $i$-th element. Also, $k_i$ in (9) represents the stiffness of a rod of cross-sectional area $A_i$, length $h$ and modulus of elasticity $E_i$. The model (17)-(20) thus describes an $n$ degree-of-freedom mass-spring system of $n$ masses, each equals to the mass of its associate rod element. They are connected by rods of length $h$, each with a stiffness corresponding to the stiffness of the appropriate rod’s element, as shown below.

The finite element model which has developed mathematically in this section can be thus obtained simply by inspection. All we need to do is to concentrate the mass of each element in the element’s center, and to connect the masses by springs with constants equal to the element stiffnesses, as shown above. Then the obtained discrete model can be analysed by the methods of chapter 4. This method of modelling has been used for many years by engineers. The model obtained is alternatively called a *lumped parameter* model.